Optimal Control and Hamilton-Jacobi Equations

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Deterministic Control System

\[
\begin{cases}
\dot{x}(t) = f(t, x(t), u(t)), & u(t) \in U \quad \text{a.e. in } [0, 1] \quad (CS) \\
x(t_0) = x_0
\end{cases}
\]

$U$ is a complete separable metric space, $t$ denotes the time

\[f : [0, 1] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \quad x_0 \in \mathbb{R}^n\]

**Controls** are Lebesgue measurable functions $u(\cdot) : [0, 1] \to U$

A trajectory of (CS) is any absolutely continuous function $x \in W^{1,1}([t_0, 1]; \mathbb{R}^n)$ satisfying for some control $u(\cdot)$

\[
\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. in } [0,1]
\]

We assume that $f(t, x, \cdot)$ is continuous $f(\cdot, x, u)$ is measurable, $f(t, x, U)$ are closed and $\exists \gamma : [0, 1] \to \mathbb{R}^+$ integrable such that

\[
\sup_{u \in U} |f(t, x, u)| \leq \gamma(t)(1 + |x|)
\]
Deterministic Control System

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Deterministic Control System

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A **trajectory** of \((CS)\) is any absolutely continuous function \( x \in W^{1,1}([t_0, 1]; \mathbb{R}^n) \) satisfying for some control \( u(\cdot) \)

\[ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. in } [0,1] \]

We assume that \( f(t, x, \cdot) \) is continuous \( f(\cdot, x, u) \) is measurable, \( f(t, x, U) \) are closed and \( \exists \gamma : [0, 1] \to \mathbb{R}_+ \) integrable such that \( \sup_{u \in U} |f(t, x, u)| \leq \gamma(t)(1 + |x|) \)
$X$ is a separable Banach space. Consider the densely defined unbounded linear operator $A$ - the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$, 

$$f : [0, 1] \times X \times U \to X, \; x_0 \in X$$

and the semilinear control system

$$\dot{x}(t) = Ax + f(t, x(t), u(t)), \; u(t) \in U, \; x(t_0) = x_0$$

Its mild trajectory is defined by

$$x(t) = S(t - t_0) x_0 + \int_{t_0}^{t} S(t - s) f(s, x(s), u(s)) \, ds \quad \forall \, t \in [t_0, 1]$$

Many of the deterministic results were already adapted to the framework of semilinear control systems (controlled PDEs). Naturally this means some assumptions on semigroups: $S(\cdot)$ has to be compact to prove existence of optimal controls.
The concept of optimal control can be described as the process of influencing the behavior of a dynamical system so as to achieve the desired goal: to maximize a profit, to minimize the energy, to get from one point to another one, etc.

“After correctly stating the problem of optimal control and having at hand some satisfactory existence theorems, augmented by necessary conditions for optimality, we can consider that we have sufficiently substantial basis to study some special problems, as for instance Moon Flight Problem”.

Value Function of the Mayer Problem

\( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \xi_0 \in \mathbb{R}^n \). Consider the Mayer’s problem:

\[
\min \left\{ g(x(1)) \mid x \text{ is a trajectory of (CS), } x(0) = \xi_0 \right\}
\]

The value function associated with this problem is defined by:

\[
\forall (t_0, x_0) \in [0, 1] \times \mathbb{R}^n
\]

\[
V(t_0, x_0) = \inf \{ g(x(1)) \mid x \text{ is a trajectory of (CS), } x(t_0) = x_0 \}
\]

\[
V(1, \cdot) = g(\cdot). \text{ In general } V \text{ is nonsmooth even for smooth data.}
\]

If \( x(\cdot) \) is a trajectory of (CS), then for any \( t_0 \leq t_1 \leq t_2 \leq 1 \),

\[
V(t_1, x(t_1)) \leq V(t_2, x(t_2)).
\]

\( \bar{x}(\cdot) \) is optimal if and only if \( V(t, \bar{x}(t)) \equiv g(\bar{x}(1)) \).

**Dynamic Programming Principle:** \( \forall h > 0 \) such that \( t_0 + h \leq 1 \)

\[
V(t_0, x_0) = \inf \{ V(t+h, x(t+h)) \mid x \text{ is a trajectory of (CS), } x(t_0) = x_0 \}
\]
The **Hamiltonian** $H : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$H(t, x, p) = \max_{u \in U} \langle p, f(t, x, u) \rangle$$

If $V \in C^1$ it satisfies the **Hamilton-Jacobi equation**

$$-V_t(t, x) + H(t, x, -V_x(t, x)) = 0, \quad V(1, \cdot) = g(\cdot)$$

Optimal (feedback) control $u(t, x) \in U$ is chosen by:

$$\langle -V_x(t, x), f(t, x, u(t, x)) \rangle = H(t, x, -V_x(t, x))$$

If $u(t, \cdot)$ is Lipschitz, then the solution $\bar{x}(\cdot)$ of

$$\dot{x}(t) = f(t, x(t), u(t, x)) \text{ a.e. in } [0, 1], \quad x(0) = \xi_0$$

is optimal for the Mayer problem.
Even if data are smooth, $V$ is not differentiable.
Active Mathematical Domains Motivated by Optimal Control

- Nonsmooth, Set-Valued Analysis, Variational Analysis (since 1975)
- Solutions of HJB equations: viscosity and bilateral (since 1983)
- Control under state constraints (since 2000)
- First order necessary optimality conditions (since 1957)
- Second order necessary optimality conditions (since 1965)
- Sensitivity relations in control (since 1986)
- .....
Outline

1. Value Function of the Mayer Problem
2. Hamilton-Jacobi-Bellman Equation
3. State Constraints
Existence of Optimal Controls

Below we always assume that for a.e. \( t \) and \( \forall r > 0, f(t, \cdot, u) \) is \( c_r(t) \)-Lipschitz on \( B(0, r) \) \( \forall u \in U \), with integrable \( c_r : [0, 1] \rightarrow \mathbb{R} \).

For the Mayer problem let \( (x_i, u_i) \) be a minimizing sequence of trajectory-control pairs. By Gronwall’s Lemma, \( \{\|x_i\|_{\infty}\} \) is bounded and \( |\dot{x}_i(t)| \leq \gamma(t)(\|x_i\|_{\infty} + 1) \).

Take a subsequence \( \{x_{i,j}\} \) converging uniformly to some \( \bar{x} : [0, 1] \rightarrow \mathbb{R}^n \) with \( \dot{x}_{i,j} \) converging weakly in \( L^1([0, 1]; \mathbb{R}^n) \) to some \( \bar{y} : [0, 1] \rightarrow \mathbb{R}^n \). Then \( \dot{x} = \bar{y}, \bar{x}(0) = \xi_0 \) and

\[
\dot{x}(t) \in \text{conv} \ f(t, \bar{x}(t), U) \ \text{a.e.}
\]

where \( \text{conv} \) denotes convex hull.

If \( f(t, x, U) \) is convex, then, by the measurable selection theorem, there exists a control \( \bar{u} \) such that \( \dot{x}(t) = f(t, \bar{x}(t), \bar{u}(t)) \) a.e.
Existence of Optimal Controls

If $g$ is lower semicontinuous, then

$$\liminf_{j \to \infty} g(x_{ij}(1)) \geq g(\bar{x}(1))$$

and therefore $\bar{u}$ is optimal control.

**Theorem**

If $f(t, x, U)$ are convex and $g$ is lower semicontinuous, then for the Mayer problem an optimal solution does exist.

For semilinear control systems only a part of this proof applies and one has either to assume, that $S(t)$ is compact for all $t > 0$ or that

$$\{(x, f(t, x, u)) \mid u \in U, x \in X\}$$

is convex. This last assumption is very strong.

For nonlinear stochastic control systems the convexity assumptions are worse even in the finite dimensional framework.
\[ \dot{x}(t) \in \text{conv } f(t, x(t), U) \text{ a.e. in } [t_0, 1] \]  

(RS)

has more \( W^{1,1}([t_0, 1]; \mathbb{R}^n) \) solutions (relaxed trajectories) than the control system (CS) for the same initial condition \( x(t_0) = x_0 \). 

**Theorem**

Let \( \bar{x} : [t_0, 1] \to \mathbb{R}^n \) be a relaxed trajectory. Then for every \( \varepsilon > 0 \) there exists a trajectory \( x \) of (CS) such that \( \|x - \bar{x}\|_{\infty} \leq \varepsilon \) and \( x(t_0) = \bar{x}(t_0) \). 

**Corollary**

If \( g \) is continuous, then the infimum in the Mayer problem is attained by a relaxed trajectory and \( V = V^{\text{rel}} \), where 

\[
V^{\text{rel}}(t_0, x_0) = \min \{ g(x(1)) \mid x \in W^{1,1} \text{ satisfies } (RS), \ x(t_0) = x_0 \} 
\]
Regularity of Value Function

Under our assumptions, sets of solutions of (CS) and (RS) depend on initial state in a locally Lipschitz way. Hence

**Theorem**

- If $\forall x \in \mathbb{R}^n$, $f(t, x, U)$ is convex and $g$ is lower semicontinuous, then $V$ is lower semicontinuous.
- If $g$ is continuous, then $V$ is continuous.
- If $g$ is locally Lipschitz, then $V(t, \cdot)$ is locally Lipschitz for every $t \in [0, 1]$. Furthermore if $\gamma$ is essentially bounded, then $V$ is locally Lipschitz.
- If $g \in C^1$, $f(t, \cdot, u) \in C^1$, $H(t, \cdot, \cdot) \in C^2$ on $\mathbb{R}^n \times (\mathbb{R}^n\setminus\{0\})$, $\nabla g$ never vanishes and for every initial datum the optimal solution of the relaxed problem is unique, then $V \in C^1$. 

Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\} \) and \( x_0 \in \mathbb{R}^n \) be such that \( \varphi(x_0) \neq \pm \infty \).

The **Fréchet superdifferential** of \( \varphi \) at \( x_0 \) is the closed convex set

\[
\partial^+ \varphi(x_0) = \{ p \in \mathbb{R}^n | \limsup_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \leq 0 \}
\]

The **Fréchet subdifferential** of \( \varphi \) at \( x_0 \) is the set:

\[
\partial^- \varphi(x_0) = \{ p \in \mathbb{R}^n | \liminf_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \geq 0 \}
\]

We always have \( \partial^+ \varphi(x_0) = -\partial^- (-\varphi)(x_0) \).
The contingent epiderivative of $\varphi$ at $x_0$ in the direction $v$ is

$$D_{\uparrow} \varphi(x_0)(v) = \lim_{\varepsilon \to 0^+, v' \to v} \inf \frac{\varphi(x_0 + \varepsilon v') - \varphi(x_0)}{\varepsilon}$$

and the contingent hypodervative of $\varphi$ at $x_0$ in the direction $v$

$$D_{\downarrow} \varphi(x_0)(v) = \lim_{\varepsilon \to 0^+, v' \to v} \sup \frac{\varphi(x_0 + \varepsilon v') - \varphi(x_0)}{\varepsilon}$$

We always have

$$\partial_- \varphi(x_0) = \{ p \in \mathbb{R}^n \mid D_{\uparrow} \varphi(x_0)(v) \geq \langle p, v \rangle \ \forall \ v \in \mathbb{R}^n \}$$

$$\partial_+ \varphi(x_0) = \{ p \in \mathbb{R}^n \mid D_{\downarrow} \varphi(x_0)(v) \leq \langle p, v \rangle \ \forall \ v \in \mathbb{R}^n \}$$
Theorem

Assume that $g$ is locally Lipschitz, $0 \leq t_0 < 1$. Consider a trajectory $z$ of (CS). If for a.e. $t \in [t_0, 1]$, $\exists p(t) \in \mathbb{R}^n$ such that

$$
(\langle p(t), \dot{z}(t) \rangle, -p(t)) \in \partial_+ V(t, z(t))
$$

then $z$ is optimal at $(t_0, z(t_0))$.

Proof — $\psi(t) := V(t, z(t))$ is absolutely continuous. Let $t \in [t_0, 1]$ be such that $\dot{\psi}(t)$ and $\dot{z}(t)$ do exist and our assumption is verified. Then $\dot{\psi}(t) \geq 0$ and

$$
0 = \langle((\langle p(t), \dot{z}(t) \rangle, -p(t)), (1, \dot{z}(t))) \geq D_{\downarrow} V(t, z(t))(1, \dot{z}(t))
\geq \limsup_{\varepsilon \to 0^+} \frac{V(t + \varepsilon, z(t + \varepsilon)) - V(t, z(t))}{\varepsilon} = \dot{\psi}(t)
$$

Thus $\dot{\psi}(t) = 0$ and therefore $t \mapsto V(t, z(t))$ is constant.
Theorem

Assume \( g \) is lsc, \( f \) is \textit{continuous} in time uniformly in \( u \) and \( f(t, x, U) \) are convex. Then for all \((t_0, x_0) \in \text{Dom}(V),\)

\[
\begin{align*}
(i) \quad t_0 < 1 & \implies \inf_{u \in U} D_\uparrow V(t_0, x_0)(1, f(t_0, x_0, u)) \leq 0 \\
(ii) \quad t_0 > 0 & \implies \sup_{u \in U} D_\uparrow V(t_0, x_0)(-1, -f(t_0, x_0, u)) \leq 0 \\
(iii) \quad t_0 < 1 & \implies \inf_{u \in U} D_\downarrow V(t_0, x_0)(1, f(t_0, x_0, u)) \geq 0
\end{align*}
\]

Proof — Consider a trajectory \( x \) of (CS) such that \( x(t_0) = x_0 \)
\( V(t, x(t)) \equiv g(x(1)) \) and \( \varepsilon_i \rightarrow 0^+ \) such that \( \frac{x(t_0 + \varepsilon_i) - x(t_0)}{\varepsilon_i} \rightarrow v. \)
Then, by convexity, \( v \in f(t_0, x_0, U). \) Therefore
\( D_\uparrow V(t_0, x_0)(1, v) \leq 0. \) Fix \( u_0 \in U \) and consider the solution of
(CS) with constant control \( u_0. \) Then for \( \varepsilon > 0 \)
\( V(t_0 - \varepsilon, x(t_0 - \varepsilon)) - V(t_0, x_0) \leq 0 \leq V(t_0 + \varepsilon, x(t_0 + \varepsilon)) - V(t_0, x_0). \)
Divide by \( \varepsilon \) and take lower and upper limits.
Corollary (Bilateral Solution)

LSC value function satisfies \( V(1, \cdot) = g, \forall (t, x) \in [0, 1[ \times \mathbb{R}^n, \)

\[
\begin{align*}
\forall (p_t, p_x) & \in \partial_- V(t, x), -p_t + H(t, x, -p_x) = 0 \\
\forall x & \in \mathbb{R}^n, \quad V(0, x) = \liminf_{t \rightarrow 0+, x \rightarrow x} V(t, x) \\
\forall x & \in \mathbb{R}^n, \quad V(1, x) = \liminf_{t \rightarrow 1-, x \rightarrow x} V(t, x)
\end{align*}
\]

Corollary (Viscosity Solution)

Continuous value function satisfies \( V(1, \cdot) = g, \forall t \in ]0, 1[, x \)

\[
\begin{align*}
\forall (p_t, p_x) & \in \partial_- V(t, x), -p_t + H(t, x, -p_x) \geq 0 \\
\forall (t, p_t, p_x) & \in \partial_+ V(t, x), -p_t + H(x, -p_x) \leq 0
\end{align*}
\]

If a function \( W \) satisfies conditions of either corollary, then it is equal to the value function.
Uniqueness of Solution to (HJB)

Steps of the proof of uniqueness

- Deduce from inequalities involving Fréchet super/sub differentials contingent inequalities. This step is not simple because $D_{\uparrow}V(t_0, x_0)(\cdot, \cdot)$ and $D_{\downarrow}V(t_0, x_0)(\cdot, \cdot)$ are not convex functions. It requires lots of nonsmooth analysis results.

- Show that any function $W$ satisfying contingent inequalities is nondecreasing along trajectories of (CS). Use invariance theorems.

- Show that for any function $W$ satisfying contingent inequalities and any $(t_0, x_0) \in dom(W)$, there exists a trajectory $x(\cdot)$ of (CS) such that $W(t, x(t))$ is nonincreasing. Use the viability theorem.

- These two monotonicity properties imply $W = V$
For a given closed set \( K \subset \mathbb{R}^n \), state constraints are expressed by

\[
    x(t) \in K \quad \text{for all} \quad t \in [0, 1]
\]

A trajectory \( x(\cdot) \) as above is called a \textbf{viable} (or feasible) trajectory of the control system. \( S_K(x_0) \) denotes set of all viable trajectories defined on \([0, 1]\) starting at \( x_0 \) at time 0.

\[
\begin{align*}
    x' &= f(x, u), \quad u \in U = \{u_1, u_2, u_3\} \\
    x(0) &= a_0, \quad b_0, \quad c_0, \quad d_0
\end{align*}
\]

\( f(a_0, u_1), f(b_0, u_2), f(c_0, u_2) \) are tangent to \( K \) at \( a_0, b_0 \) and \( c_0 \). At \( d_0 \) there is no \( u \in U \) such that \( f(d_0, u) \) is tangent to \( K \).
Example: Milyutin, 2000 (also for higher order)

\[
\begin{align*}
\min \int_0^T \left( x(t) + \frac{|u(t)|^p}{p} \right) dt, & \quad p > 1 \\
x'''(t) = u(t), & \quad x(0) = \xi_0, \ x'(0) = \xi_1, \ x''(0) = \xi_2 \\
u(t) \in \mathbb{R}, & \quad x(t) \geq 0
\end{align*}
\]

If an initial condition \((\xi_0, \xi_1, \xi_2)\) is admissible, then the optimal solution exists and is unique.

For “most” of the admissible initial conditions \((\xi_0, \xi_1, \xi_2)\) there exist \(T = T(\xi_0, \xi_1, \xi_2) > 0\) such that the optimal trajectory reaches the boundary of constraints an infinite (countable) number of times with \(T\) being their accumulation point.
Neighboring Feasible Trajectories (NFT) Estimates

**Distance** from $z \in \mathbb{R}^n$ to $K$, $d_K(z) := \min\{|z - y| : y \in K\}$.

Can we “control” trajectories violating state constraints?

What are sufficient conditions for the following property:

For every trajectory $\hat{x}(\cdot)$ of (CS) with $\hat{x}(0) \in K$ there exists a **viable trajectory** $x(\cdot) \in S_K(\hat{x}(0))$ satisfying the following NFT estimates in $\|\cdot\|_{W^{1,1}}$

$$\|x - \hat{x}\|_{W^{1,1}} \leq C \max_{t \in [0,1]} d_K(\hat{x}(t))$$

or the NFT estimates in $\|\cdot\|_{\infty}$

$$\|x - \hat{x}\|_{\infty} \leq C \max_{t \in [0,1]} d_K(\hat{x}(t))$$

where $C$ depends on the magnitude of $|\hat{x}(0)|$, but **not** on $\hat{x}(\cdot)$?
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Neighboring Feasible Trajectories (NFT) Estimates

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For every trajectory \( \hat{x}(\cdot) \) of (CS) with \( \hat{x}(0) \in K \) there exists a viable trajectory \( x(\cdot) \in S_K(\hat{x}(0)) \) satisfying the following NFT estimates in \( \|\cdot\|_{W^{1,1}} \)

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\|x - \hat{x}\|_{W^{1,1}} \leq C \max_{t \in [0,1]} d_K(\hat{x}(t))
\]

or the NFT estimates in \( \|\cdot\|_{\infty} \)

\[
\|x - \hat{x}\|_{\infty} \leq C \max_{t \in [0,1]} d_K(\hat{x}(t))
\]

where \( C \) depends on the magnitude of \( |\hat{x}(0)| \), but not on \( \hat{x}(\cdot) \)?
\[ \|x - \hat{x}\|_{W^{1,1}} \leq C \max \{ d_K(\hat{x}(t)) : t \in [0,1] \} \]
Let $U$ be compact, $f$ be time independent, and $\exists \ k \geq 0$ such that $f(\cdot, u)$ is $k$-Lipschitz (uniformly in $u$).

Let $K$ be the closure of an open set with $C^{1,1}$ boundary $\partial K$ and assume the inward pointing condition (Soner, 1986) : $\forall \ x \in \partial K$

$$\min_{u \in U} \langle n_x, f(x, u) \rangle < 0,$$

where $n_x$ is the **outward unit normal** to $K$ at $x$

Then NFT estimates in $\| \cdot \|_\infty$ hold true.

Can be extended to $f$ measurable in time, by adding an inward pointing condition on a neighborhood of $\partial K$. 
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Optimal Control
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Tangent and Normal Cones

Let \((X, d)\) be a metric space \(A_\tau \subset X, \ \tau \in \mathcal{T}\).

\[
\text{Limsup}_{\tau \to \tau_0} A_\tau = \{ v \in X \mid \liminf_{\tau \to \tau_0} d_{A_\tau}(v) = 0 \}
\]

\[
\text{Liminf}_{\tau \to \tau_0} A_\tau = \{ v \in X \mid \limsup_{\tau \to \tau_0} d_{A_\tau}(v) = 0 \}
\]

(the Peano-Kuratowski upper and lower limits)

The **contingent cone** \(T_K(x)\) to \(K \subset \mathbb{R}^n\) at \(x \in K\) is the set of \(v \in \mathbb{R}^n\) such that \(\exists \varepsilon_i \to 0+, \ v_i \to v\) satisfying \(x + \varepsilon_i v_i \in K\)

\[
T_K(x) = \text{Limsup}_{\varepsilon \to 0+} \frac{K - x}{\varepsilon}
\]

The **limiting normal cone** to a closed subset \(K \subset \mathbb{R}^n\) at \(x \in K\) is

\[
N^L_K(x) = \text{Limsup}_{y \to_K x}[T_K(y)]^-
\]

\(\to_K\) stands for the convergence in \(K\) and

\[
T_K(y)^- = \{ p \in \mathbb{R}^n \mid \langle p, v \rangle \leq 0 \ \forall \ v \in T_K(y) \}\]
The Clarke normal cone to $K$ at $x \in K$

$$N_K(x) := \text{conv} \ N^L_K(x)$$

Normalized normals: $N^1_K(x) := N_K(x) \cap S^{n-1}$

**Generalized Inward Pointing Condition** (HF, Rampazzo, 2000)

$$\min_{u \in U} \max_{p \in N^1_K(x)} \langle p, f(t, x, u) \rangle < 0 \quad \forall x \in \partial K$$

Assume $f(\cdot, \cdot, u)$ is locally Lipschitz (uniformly in $u$). Then NFT estimates in $\| \cdot \|_{\infty}$ hold true.

A counterexample to NFT estimates in $\| \cdot \|_{W^{1,1}}$ was given in P. Bettiol, A. Bressan & R. Vinter, SICON (2010) for $f$ independent from $t$, $x$ and $K$ being an intersection of two half spaces in $\mathbb{R}^2$. 
General Set \( K \)

The **Clarke normal** cone to \( K \) at \( x \in K \)

\[
N_K(x) := \overline{\text{conv}} N^L_K(x)
\]

Normalized normals: \( N^1_K(x) := N_K(x) \cap S^{n-1} \)

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Other Counterexamples

and a counterexample to NFT estimates in $\| \cdot \|_\infty$ when $K$ is an intersection of two half spaces in $\mathbb{R}^3$ and $f$ is independent from $x$ and is measurable in time. May we expect a weaker logarithmic estimate when $\max_{t \in [0,1]} d_K(\hat{x}(t)) > 0$?

$$\| x - \hat{x} \|_\infty \leq C \max_{t \in [0,1]} d_K(\hat{x}(t)) \left| \log \left( \max_{t \in [0,1]} d_K(\hat{x}(t)) \right) \right|$$

or the Hölder estimate: for some $\alpha \in (0, 1)$

$$\| x - \hat{x} \|_\infty \leq C \max_{t \in [0,1]} d_K(\hat{x}(t))^\alpha$$

There are a counterexample to the logarithmic estimates for $f$ independent from $x$ and continuous in time and a counterexample to Hölder estimates for $f$ independent from $x$ and measurable in time.

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There are a counterexample to the logarithmic estimates for $f$ independent from $x$ and continuous in time and a counterexample to Hölder estimates for $f$ independent from $x$ and measurable in time.
Counterexample (Bettiol, Bressan, Vinter)

\[ f(t, x, U) = G := \text{co} \{ (2, 1), (-2, 1), (0, 0) \} \]

\[ K = \{ (x_1, x_2) : x_2 \geq |x_1| \} \]
Trajectory Violating State Constraints

\( \hat{x}(t) = (\hat{x}_1(t), t) \)
Getting Rid of this Counterexample

\[ G_1 = G \cup \{(2, 3), (-2, 3)\} \]

Then NFT estimates in \( W^{1,1} \) hold true.
Revisiting Classical Condition

Let $K$ be the closure of an open set with $C^2$ boundary $\partial K$ and assume the inward pointing condition: $\forall x \in \partial K$, $\exists u_x \in U$ with $\langle n_x, f(x, u_x) \rangle < 0$. Then for every $u \in U$ with $\langle n_x, f(x, u) \rangle \geq 0$

$$\langle n_x, f(x, u_x) - f(x, u) \rangle < 0$$

The generalization of this last inward pointing condition is:

$$\forall t \in [0, 1], \forall x \in \partial K,$$

$$(IPC) \quad \left\{ \begin{array}{l}
\forall \nu \in U \text{ with } \max_{n \in N^1_K(x)} \langle n, f(t, x, \nu) \rangle \geq 0 \\
\exists w \in \operatorname{Liminf}{(s, y) \rightarrow (t, x)} f(s, y, U) \\
\max_{n \in N^1_K(x)} \langle n, w - f(t, x, \nu) \rangle < 0
\end{array} \right.$$

The relaxed inward pointing condition: $\forall t \in [0, 1], \forall x \in \partial K,$

$$(IPC_{rel}) \quad \text{same as (IPC) but } w \in \operatorname{Liminf}{(s, y) \rightarrow (t, x)} \operatorname{conv} f(s, y, U)$$
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$$(IPC) \begin{cases} \forall \, v \in U \noindent \text{with} \noindent \max_{n \in N^1_K(x)} \langle n, f(t, x, v) \rangle \geq 0 \\ \exists \, w \in \text{Liminf}_{(s,y) \rightarrow (t,x)} f(s, y, U) \\ \max_{n \in N^1_K(x)} \langle n, w - f(t, x, v) \rangle < 0 \end{cases}$$

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The generalization of this last inward pointing condition is:

$\forall t \in [0, 1], \forall x \in \partial K$, $\forall \nu \in U$ with $\max_{n \in N_K^1(x)} \langle n, f(t, x, \nu) \rangle \geq 0$

$\exists w \in \text{Liminf}_{(s,y) \to (t,x)} f(s, y, U)$

$\max_{n \in N_K^1(x)} \langle n, w - f(t, x, \nu) \rangle < 0$

The relaxed inward pointing condition: $\forall t \in [0, 1], \forall x \in \partial K$, $\forall \nu \in U$ with $\max_{n \in N_K^1(x)} \langle n, f(t, x, \nu) \rangle \geq 0$

$\exists w \in \text{Liminf}_{(s,y) \to (t,x)} \text{conv} f(s, y, U)$
For $f(\cdot, \cdot, U)$ having a Closed Graph

**Theorem (NODEA, 2013)**

Assume that $f(\cdot, \cdot, U)$ has closed graph and (IPC). Then NFT estimates in $\| \cdot \|_{W^{1,1}}$ hold true.

If $f$ is measurable in time, then a similar result is valid but a uniform inward pointing condition has to be imposed on a neighborhood of the boundary of $K$.

**Theorem (NODEA, 2013)**

Assume that $f(\cdot, \cdot, U)$ has closed graph and (IPC$_{rel}$). Then for any viable relaxed trajectory $x^{rel}(\cdot)$ and every $\delta > 0$ there exists a trajectory $x(\cdot)$ of (CS) such that $x(0) = x^{rel}(0)$, $x(t) \in \text{Int } K \ \forall \ t \in (0, 1]$ and $\|x(\cdot) - \hat{x}(\cdot)\|_{\infty} \leq \delta$. 
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\[
x(0) = x^{\text{rel}}(0), \quad x(t) \in \text{Int} \ K \quad \forall \ t \in (0, 1] \quad \text{and} \quad \| x(\cdot) - \hat{x}(\cdot) \|_{\infty} \leq \delta
\]
Relaxed Semilinear Control System

\[ \dot{x}(t) \in A x + \text{conv } f(t, x(t), U), \quad x(t_0) = x_0 \]

Its **mild trajectory** is defined by

\[ x(t) = S(t - t_0) x_0 + \int_{t_0}^{t} S(t - s) \nu(s) \, ds \quad \forall \ t \in [t_0, 1] \]

where \( \nu(s) \in \text{conv } f(s, x(s), U) \).

Assume \( X \) is Hilbert and \( K = \overline{\text{Int } K} \subset X \). Denote by \( Z \) the set of points \( z \in X \setminus \partial K \) admitting a unique projection \( P_{\partial K}(z) \) on \( \partial K \). For every \( z \in Z \), set

\[ n_z = \frac{z - P_{\partial K}(z)}{\|z - P_{\partial K}(z)\|_X} \quad \text{sgn}(d_{\partial K}^{\text{oriented}}(z)) \]
The analogue of relaxation theorem is valid under the following inward pointing condition:

∀ \bar{x} \in \partial K, \exists \eta, \rho, M > 0 such that ∀ t \in [0,1], ∀ x \in K \cap B(\bar{x}, \eta), ∀ v \in U satisfying sup \tau \leq \eta, z \in Z \cap B(x, \eta) \langle n_z, S(\tau) f(t, x, v) \rangle \geq 0, we have

\left\{ \bar{v} \in U : \| f(t, x, \bar{v}) - f(t, x, v) \| \leq M , \sup_{\tau \leq \eta, z \in Z \cap B(S(\tau)x, \eta)} \langle n_z, S(\tau) (f(t, x, \bar{v}) - f(t, x, v)) \rangle < -\rho \right\} \neq \emptyset .

If K is convex, then Z can be replaced by X \setminus K.
The analogue of relaxation theorem is valid under the following inward pointing condition:

\[
\forall \bar{x} \in \partial K, \exists \eta, \rho, M > 0 \text{ such that } \forall t \in [0, 1], \forall x \in K \cap B(\bar{x}, \eta),
\forall v \in U \text{ satisfying } \sup_{\tau \leq \eta, z \in Z \cap B(x, \eta)} \langle n_z, S(\tau) f(t, x, v) \rangle \geq 0, \text{ we have}
\]

\[
\left\{ \bar{v} \in U : \|f(t, x, \bar{v}) - f(t, x, v)\| \leq M, \right. \\
\left. \sup_{\tau \leq \eta, z \in Z \cap B(S(\tau)x, \eta)} \langle n_z, S(\tau)(f(t, x, \bar{v}) - f(t, x, v)) \rangle < -\rho \right\} \neq \emptyset.
\]

If \( K \) is convex, then \( Z \) can be replaced by \( X \setminus K \).
Value Function of the Constrained Mayer Problem

For \( g : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \} \) the \textbf{Mayer problem under state constraints} is

\[
\text{minimize} \ \{ g(x(1)) \mid x(\cdot) \in S_K(\xi_0) \}
\]

The \textit{value function} is defined by

\[
V(t_0, x_0) = \inf \{ g(x(1)) \mid x(\cdot) \text{ is a viable trajectory of } (CS), \ x(t_0) = x_0 \}
\]

The \textit{relaxed} Mayer problem is:

\[
\text{min} \ \{ g(x(1)) \mid x(\cdot) \text{ is a viable relaxed trajectory of } (RS), \ x(0) = \xi_0 \}
\]

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For $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ the **Mayer problem under state constraints** is

$$\text{minimize } \{g(x(1)) \mid x(\cdot) \in S_K(\xi_0)\}$$

The **value function** is defined by

$$V(t_0, x_0) = \inf \{g(x(1)) \mid x(\cdot) \text{ is a viable trajectory of (CS), } x(t_0) = x_0\}$$

The **relaxed** Mayer problem is:

$$\min \{g(x(1)) \mid x(\cdot) \text{ is a viable relaxed trajectory of (RS), } x(0) = \xi_0\}$$
Corollary (NODEA, 2013)

Assume \((\text{IPC}_{\text{rel}})\), \(f\) is continuous in \(t\), uniformly in \(u\), \(\gamma\) is essentially bounded and that \(g\) is locally Lipschitz. Then \(V\) is locally Lipschitz on \([0, 1] \times K\) and is equal to the value function of the relaxed problem. Furthermore if \(\bar{x}(\cdot)\) is an optimal solution to the Mayer problem, then it is also optimal for the relaxed problem.

Outward Pointing Condition (OPC)

\[
\begin{cases} 
  \forall \ t \in [0, 1], \ x \in \partial K, \ v \in U \text{ with } \min_{n \in N_K^1(x)} \langle n, f(t, x, v) \rangle \leq 0 \\
  \exists u \in U, \min_{n \in N_K^1(x) \cap S^{n-1}} \langle n, f(t, x, u) - f(t, x, v) \rangle \geq 0.
\end{cases}
\]
Corollary (NODEA, 2013)

Assume \((IPC_{rel})\), \(f\) is continuous in \(t\), uniformly in \(u\), \(\gamma\) is essentially bounded and that \(g\) is locally Lipschitz.

Then \(V\) is locally Lipschitz on \([0, 1] \times K\) and is equal to the value function of the relaxed problem.

Furthermore if \(\bar{x}(\cdot)\) is an optimal solution to the Mayer problem, then it is also optimal for the relaxed problem.

Outward Pointing Condition (OPC)

\[
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\end{aligned}
\]
Theorem (Uniqueness of constrained viscosity solutions, Calc. Var. & PDEs, 2013)

Let $W : [0, 1] \times K \rightarrow \mathbb{R}$. Assume (IPC) and that $g$ and $H$ are continuous and $W(1, \cdot) = g(\cdot)$. Then the following statements are equivalent:

(a) $W$ is the value function of the Mayer problem;

(b) $W$ is continuous and

(i) $-p_t + H(t, x, -p_x) \leq 0, \quad \forall (p_t, p_x) \in \partial_+ W(t, x), \forall (t, x) \in ]0, 1[ \times \text{Int } K$;

(ii) $-p_t + H(t, x, -p_x) \geq 0, \quad \forall (p_t, p_x) \in \partial_- W(t, x), \forall (t, x) \in ]0, 1[ \times K$.

In other words, $V$ is the unique continuous viscosity solution of the constrained HJB equation.
Theorem (Uniqueness of constrained bilateral solutions, Calc. Var. & PDEs, 2013)

Assume (OPC), that $H$ is continuous, $g$ is lsc and let $W : [0, 1] \times K \to \mathbb{R} \cup \{+\infty\}, W(1, \cdot) = g(\cdot)$. If $f(t, x, U)$ are convex, then the following two statements are equivalent:

(a) $W$ is the value function of the Mayer problem;

(b) $W$ is lower semicontinuous and

(i) $-p_t + H(t, x, -p_x) = 0$, $\forall (p_t, p_x) \in \partial_- W(t, x)$, $\forall (t, x) \in ]0, 1[ \times \text{Int } K$;

(ii) $-p_t + H(t, x, -p_x) \geq 0$, $\forall (p_t, p_x) \in \partial_- W(t, x)$, $\forall (t, x) \in (]0, 1[ \times \partial K) \cup (\{0\} \times K)$;

(iii) $\lim \inf (t', x') \to (1, x)$ $\quad W(t', x') = g(x)$, $\forall x \in K$.
Conclusions and Open Questions

- Value function is the unique solution of Hamilton-Jacobi-Bellman equation
- Same in the presence of state constraints
- For the controlled PDEs that can be reduced to semilinear control systems without state constraints HJB were already investigated
- Many Open Questions in : HJB associated to semilinear control systems in the presence of state constraints
- Many Open Questions in : Stochastic Optimal Control

NFT theorems for semilinear control systems - to appear in ESAIM: Control, Optimisation and Calculus of Variations